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**NATIONAL ADVISORY COMMITTEE
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TECHNICAL MEMORANDUM

No. 1174

ANALYTICAL TREATMENT OF NORMAL CONDENSATION SHOCK

By Heybey

Translation

“Analytische Behandlung des geraden Kondensationsstosses.”
Heeres-Versuchsstelle, Peenemünde, Archiv Nr. 66/72, März 30, 1942



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INTRODUCTION

The condensation of water vapor in an air stream has the following consequences:

1. Acquisition of heat (liberated heat of vaporization)
2. Loss of mass on the part of the flowing gas (water vapor is converted to liquid)
3. Change in the specific gas constants and of the ratio k of the specific heats (caused by change of gas composition)

A discontinuous change of state is therefore connected with the condensation; schlieren photographs of supersonic flows in two-dimensional Laval nozzles show two intersecting oblique shock fronts that in the case of high humidities may merge near the point of intersection into one normal shock front. The following discussion will deal with normal shock fronts only; it will be assumed that the velocity vector may be considered as being at right angles to the shock front (one-dimensional theory). All flow properties directly ahead of the shock will be designated by the subscript 1, all those directly behind the shock by the subscript 2, all those referring to a stagnation condition by the subscript o , and all those applying in the narrowest cross section of the nozzle F^* by an asterisk. [NACA comment: Particular attention is called to the fact that capital T denotes stagnation temperature only when used with the subscript o . Otherwise it denotes static temperature.] The sonic velocity will be denoted by a , and the flow velocity by w .

The following equation exists for the sum of kinetic and caloric energy (enthalpy $i = c_p T$) per unit mass of the flowing gas:

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$$\frac{w_1^2}{2} + i_1 = i_0 = \frac{1}{2} \frac{k+1}{k-1} a^{*2} \quad (1)$$

Directly behind the shock the constant total energy i_0 has been supplemented by a quantity of heat energy per unit mass q as follows:

$$\frac{w_2^2}{2} + i_2 = i_0 + q = i_0' = \frac{1}{2} \frac{k+1}{k-1} a^{*'}^2 \quad (1')$$

Because $i_0' > i_0$, for the description of the flow behind the shock a higher stagnation temperature must be specified than that which characterized the flow before the shock.

It will become apparent later that the values of p_0 , ρ_0 , and F^* must also be changed. From equations (1) and (1') for a^{*} , is obtained

$$\frac{a^{*'}}{a^{*}} \equiv Q > 1 \quad (2a)$$

as Q was thus defined as an abbreviation for $\frac{a^{*'}}{a^{*}}$, another equation may be written

$$\frac{i_0'}{i_0} = \frac{T_0'}{T_0} = Q^2 (> 1) \quad (2b)$$

From equation (2b) it follows that with $i_0' = i_0 + q$

$$Q^2 = 1 + \frac{q}{i_0} \quad (3)$$

The quantity Q (inherently positive) is clearly determined by the quantity of heat of vaporization liberated; Q increases with that quantity.

The loss in mass of flowing gas due to condensation, as well as the alteration of the gas constants and of k will be assumed to be so small that they may be disregarded. Other conditions being the same (for example, relative humidity = 100 percent), this assumption becomes more exact in proportion as T_0 is lower. If the air flows into the Laval nozzle from the free atmosphere with T_0 approximately 273°K , the gravimetric proportion of saturated water vapor is 2 to 3 percent (this proportion remains the same through all changes of state so long as these changes occur without gain or loss of water vapor).

Accordingly, the following equations apply:

$$\rho_1 w_1 = \rho_2 w_2 \quad (4)$$

$$p_1 + \rho_1 w_1^2 = p_2 + \rho_2 w_2^2 \quad (5)$$

$$\frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2} \quad (6)$$

The equations (4), (5), and (6) express the constancy of the mass flow, the momentum, and the specific gas constants R . Together with equations (1) and (1'), these equations constitute the basic equations of a simplified theory of the normal condensation shock. If the condensation occurs in a subsonic flow, the associated pressure change, because it is propagated at sonic speed, quickly spreads throughout the whole flow. The same pressure spread applies to all aspects of the change of state. No shock phenomenon can develop. Therefore, in the following discussion a fundamental assumption is that w_1 is a supersonic velocity.

CHANGE OF STATE AT SITE OF SHOCK

A suitable combination of the five basic equations leads through a rather long mathematical process, which will be omitted here, to the following relation:

$$\frac{w_2}{w_1} = \frac{w_1 w_2 - a^{*2}}{w_1 w_2 - a^{*2}} \quad (7)$$

For $a^{*'} = a^*$ ($Q = 1$) this becomes the Prandtl formula $w_1 w_2 = a^{*2}$.

By introduction of the abbreviations

$$\frac{w_1}{a^*} = x_1 \quad \text{and} \quad \frac{w_2}{a^{*'}} = x_2$$

equation (8) is obtained

$$\frac{w_2}{w_1} = Q \frac{x_2}{x_1} \quad (8)$$

From equation (7) is obtained

$$\frac{x_2}{x_1} Q = \frac{x_1 x_2 - Q}{x_1 x_2 - \frac{1}{Q}}$$

or

$$x_2^2 - x_2 \frac{1 + x_1^2}{Qx_1} + 1 = 0 \quad (7a)$$

From these equations, an equation to be used subsequently is obtained

$$Qx_1 (1 + x_2^2) = x_2 (1 + x_1^2) \quad (7b)$$

and by further conversion

$$\frac{w_2}{w_1} = \frac{w_2^2 + a^{*2}}{w_1^2 + a^{*2}} \quad (7c)$$

which is another form of equation (7).

The solution of equation (7a) defines x_2 as a function of x_1 and Q . With the aid of this equation, simple expressions for ρ_2/ρ_1 , T_2/T_1 , Ma_2/Ma_1 , and p_2/p_1 may be derived. [NACA comment: Ma , Mach number.]

From equations (4) and (3) it follows that

$$\frac{\rho_2}{\rho_1} = \frac{1}{Q} \frac{x_1}{x_2} \quad (9)$$

If the well-known relation between T/T_0 and w/a^* is applied to T_1/T_0 and T_2/T_0 , by consideration of equation (2b) also, the following equation is obtained:

$$\frac{T_2}{T_1} = Q^2 \frac{1 - \frac{k-1}{k+1} x_2^2}{1 - \frac{k-1}{k+2} x_1^2} \quad (10)$$

Because furthermore,

$$\frac{Ma_2}{Ma_1} = \frac{w_2}{w_1} \frac{a_1}{a_2} = \frac{w_2}{w_1} \sqrt{\frac{T_1}{T_2}}$$

it follows that

$$\frac{Ma_2}{Ma_1} = \frac{x_2}{x_1} \sqrt{\frac{1 - \frac{k-1}{k+1} x_1^2}{1 - \frac{k-1}{k+1} x_2^2}} \quad (11)$$

From equation (6) and with the aid of equations (9) and (10)

$$\frac{p_2}{p_1} = Q \frac{x_1}{x_2} \frac{1 - \frac{k-1}{k+1} x_2^2}{1 - \frac{k-1}{k+1} x_1^2} = \frac{1 + x_1^2 - \frac{2k}{k+1} Q x_1 x_2}{1 - \frac{k-1}{k+1} x_1^2} \quad (12)$$

The expression on the right is derived from the middle expression by the application of equation (7b).

The solution of equation (7a) is

$$x_2 = \frac{1 + x_1^2 \pm \sqrt{(1 + x_1^2)^2 - 4Q^2 x_1^2}}{2Qx_1}$$

Because x_2 , a ratio of velocities, must be a real number, from the previous equation an upper limit is obtained for Q ; the lower limit is by definition $Q = 1$. It is evidently true that

$$1 \leq Q \leq \frac{1 + x_1^2}{2x_1} \quad (13)$$

Because in the supersonic region $\frac{w_1}{a^*} > 1$, accordingly the absolute value of the root is found to be

$$(x_1^2 - 1) \geq \sqrt{(1 + x_1^2)^2 - 4Q^2 x_1^2} \geq 0$$

In order to decide the sign of the root, x_1 being fixed and Q being variable, the values x_2 may assume for each of the two signs must first be determined. For the variation of x_2 with Q it appears that

$$\frac{\partial x_2}{\partial Q} = \pm \frac{x_2 (1 + x_1^2)}{Q \sqrt{(1 + x_1^2)^2 - 4Q^2 x_1^2}}$$

where the absolute value of the root is in the denominator. With the minus sign, x_2 is a continually decreasing function of Q ,

whereby it follows from equation (13) that

$$x_1 \geq x_2 \geq 1 \quad (14a)$$

Correspondingly, with the plus sign (x_2 increasing with Q) it is found that

$$\frac{1}{x_1} \leq x_2 \leq 1 \quad (14b)$$

In the first case, the air flow directly behind the shock is always moving at supersonic velocity; in the second case, always at subsonic velocity. Only for $Q = Q_{\max} = \frac{1 + x_1^2}{2x_1}$ is exactly sonic velocity found in both cases.

The first possibility is supported by experimental findings. In the schlieren photographs, Mach waves or shock fronts both of which can arise only at supersonic velocities may be observed behind the shock or at least at a certain distance downstream. If subsonic velocity existed directly behind the shock, a subsequent constriction of the cross section of the flow filaments would have to occur (in order to produce a minimum cross section). This course of the flow is improbable because the shock always occurs at a point where the walls diverge downstream. Therefore the positive sign must be chosen for the root

$$x_2 = \frac{1 + x_1^2 + \sqrt{(1 + x_1^2)^2 - 4Q^2x_1^2}}{2Qx_1} \quad (7d)$$

and equation (14a) is obtained. The normal condensation shock effects a change from one supersonic velocity to another supersonic velocity. The nomograph (fig. 1) shows the relation between Q , x_1 , and x_2 as determined by equation (7d). In regard to the end point of the Q scale, see the remarks at the end of this section.

After the sign of the root has been determined, $x_2 = w_2/a^*$ is clearly a function of x_1 and Q . The same relation is therefore true of the quotients in equations (8) to (12). No change is effected in this respect if the quotients are multiplied by x_1 , ρ_1/ρ_0 , T_1/T_0 , Ma_1 , or p_1/p_0 because these quantities are also clearly functions of $x_1 = w_1/a^*$. Because the quantity x_1 is in turn clearly a function of the Mach number Ma_1 immediately ahead of the shock and Q is clearly a function of q in accordance with equation (3), the following principle is obtained:

Pressure, density, temperature, Mach number, and velocity immediately behind a normal condensation shock are clearly determined by the Mach number at which the shock occurs and by the quantity of heat liberated.

Thus both these quantities must be known if determinations of the conditions directly behind the shock are desired (assuming that P_0 , ρ_0 , T_0 , and a^* are known). Instead of the Mach number, the given quantity may, of course, be w_1/a^* , p_1/p_0 , ρ_1/ρ_0 , T_1/T_0 , or F^*/F because in the supersonic region all these quantities may be expressed as functions of each other.

At what Mach number (or temperature) the supercooled vapor condenses, in what quantity the water is precipitated, and how much heat is accordingly supplied to the air stream, are questions that the resources of gas dynamics are insufficient to answer; hence, Ma_1 and q must be regarded here as variables that are independent of one another. However, in accordance with equation (13), there is for every value of x_1 (or Ma_1) a maximum for Q , and hence also for q . Therefore, if the condensation occurs at a certain Mach number Ma_1 , the liberation of more heat than a certain maximum determined by Ma_1 is impossible. From equations (13) and (3) is obtained

$$0 \leq q \leq i_0 \frac{(x_1^2 - 1)^2}{4x_1^2} \quad (13a)$$

For $x_1 = 1$, $q = 0$. Hence the normal condensation shock never occurs in the narrowest cross section of the nozzle; supersonic velocity must first have been attained.

With consideration of equations (13) and (14a), the limits may be determined between which, with a given value of x_1 (that is, with a given site of the shock), the quotients in equations (8) to (12) must lie. From what has been said in connection with equation (14a), it is evident that x_2 decreases as Q increases. In accord with equation (8), the same relation is also true for w_2/w_1 , whereby

$$1 \leq \frac{w_2}{w_1} \leq \frac{1 + x_1^2}{2x_1^2} \quad (8a)$$

In accordance with equation (4), the quotient ρ_2/ρ_1 acts in the contrary manner

$$1 \leq \frac{\rho_2}{\rho_1} \leq \frac{2x_1^2}{1+x_1^2} \quad (9a)$$

It is evident from equation (10) that T_2/T_1 increases simultaneously with Q ; that is

$$1 \leq \frac{T_2}{T_1} \leq \frac{2}{k+1} \frac{(1+x_1^2)^2}{4x_1^2 \left(1 - \frac{k-1}{k+1} x_1^2\right)} \quad (10a)$$

The ratio $\frac{Ma_2}{Ma_1} = \frac{w_2}{w_1} \sqrt{\frac{T_1}{T_2}}$ decreases with increasing Q (because w_2/w_1 and T_1/T_2 decrease).

$$1 \geq \frac{Ma_2}{Ma_1} \geq \sqrt{\frac{\frac{k+1}{2} - \frac{k-1}{2} x_1^2}{x_1^2}} = \frac{1}{Ma_1} \quad (11a)$$

Because ρ_2/ρ_1 and T_2/T_1 increase, p_2/p_1 also must increase in accordance with equation (6). If $Q = Q_{\max}$, the root in equation (7d) is eliminated and the equation becomes:

$$2Q_{\max} x_1 x_2 = 1 + x_1^2$$

From equation (12) is derived

$$1 \leq \frac{p_2}{p_1} \leq \frac{1}{k+1} \frac{1+x_1^2}{1 - \frac{k-1}{k+1} x_1^2} \quad (12a)$$

The limits in the left side of equations (8a) to (12a) are valid for $Q = 1$, that is, for the case in which no condensation shock is present at all. It is evident that:

The normal condensation shock effects an increase in pressure, density, and temperature, whereas velocity and Mach number diminish.

The ordinary normal compression shock, which occurs when a flow meets superior pressure, has these same characteristics. Nevertheless, the compression shock must be regarded as something different in principle from the condensation shock, which is initiated by the introduction of a quantity of heat. The difference is shown by the fact, among others, that the compression shock always produces subsonic

velocity. Also in such a case the ratio $w_2/w_1 = 1/x_1^2$ decreases with increasing x_1 , whereas in the case of the condensation shock it increases, as may be seen from the derivative that may be obtained from equation (7d):

$$\frac{\partial \left(\frac{w_2}{w_1} \right)}{\partial x_1} = Q \frac{\partial \left(\frac{x_2}{x_1} \right)}{\partial x_1} = \frac{2Q (Qx_1 - x_2)}{x_1^2 \sqrt{(1 + x_1^2)^2 - 4Q^2 x_1^2}} > 0$$

because

$$Q > 1 \quad \text{and} \quad x_1 > x_2$$

The same statements apply for the absolute value of the velocity w_2 . In regard to the compression shock, this follows from the Prandtl equation $w_2 = a^*/x_1$; in regard to the condensation shock, from the fact that if x_2/x_1 increases with x_1 , x_2 and therefore $w_2 = a^*x_2$ must also increase. The quotient ρ_2/ρ_1 acts in the opposite manner.

In the case in which $\frac{w_1}{a^*} = \sqrt{\frac{k+1}{k-1}}$ and $p_1 = \rho_1 = T_1 = 0$, the formula presented in equation (7) is inapplicable. For as soon as this extreme condition is approached, the concepts and therefore the equations of continuity that were used in the derivation of equation (7) no longer apply.

If this inapplication is disregarded for the moment, equation (7) is valid for all shocks occurring at finite nozzle cross sections ($x_1 < \sqrt{\frac{k+1}{k-1}}$). The condition expressed in equation (13) in conjunction with equation (7) gives

$$Q_{\max} = \frac{1}{2} \left(\frac{1}{x_1} + x_1 \right)$$

a function that increases with x_1 if $x_1 > 1$. If the quantity x_1 is allowed to approach its maximum value $\sqrt{\frac{k+1}{k-1}}$, an (unattainable) absolute upper limit for Q is obtained as follows

$$Q_{\max \max} = \frac{k}{\sqrt{k^2 - 1}} \approx 1.424$$

To this value there corresponds an absolute maximum value for the quantity of heat liberated; from equation (13a) is obtained

$$q_{\max \max} = \frac{i_0}{k^2 - 1} \approx 1.027 i_0$$

Thus no more heat than this can be added by a normal condensation shock under any circumstances. If for example $T_0 = 273^\circ \text{C}$, with $c_p = 0.238 \frac{\text{calorie}}{\text{gram degree}}$ the following is obtained:

$$q_{\max \max} \approx 66.7 \frac{\text{calorie}}{\text{gram}}$$

In figure 2, Q_{\max} and in figure 3, (q/i_0) are shown as functions of x_1 .

RATIOS p_0'/p_0 , ρ_0'/ρ_0 , AND F^*/F^*

Increase of Entropy

It was pointed out in the Introduction that for the description of the flow behind the normal condensation shock a stagnation temperature $T_0' > T_0$ must be specified. In the following discussion, it will be shown, first by means of a special case, that F^* , ρ_0 , and p_0 also must be altered.

It follows from equation (11a) that for $Q = Q_{\max}$ the following equation applies

$$Ma_2 = (Ma_2)_{\min} = 1$$

In this special case, exactly sonic velocity occurs directly behind the shock. Because this occurrence, however, is always present in the narrowest cross section of a flow, the shock cross section $F_S > F^*$ is therefore to be regarded as the minimum cross section with respect to the flow behind the shock.

On the same basis, the relations must be written in this case

$$\rho_2 = \rho^{*'} \quad \text{and} \quad p_2 = p^{*'}$$

From equations (9a) and (12a) it also follows, that if ρ_1/ρ_0 and

p_1/p_0 are expressed in terms of x_1 according to known procedure and if the expressions for ρ^*/ρ_0 and p^*/p_0 are introduced

$$\rho^{*'} = \left(\frac{k+1}{2}\right)^{\frac{1}{k-1}} \frac{2x_1^2 \left(1 - \frac{k-1}{k+1} x_1^2\right)^{\frac{1}{k-1}}}{1 + x_1^2} \rho^* \neq \rho^*$$

$$p^{*'} = \frac{1}{k+1} \left(\frac{k+1}{2}\right)^{\frac{k}{k-1}} \left(1 - \frac{k-1}{k+1} x_1^2\right)^{\frac{1}{k-1}} (1 + x_1^2) p^* \neq p^*$$

Because a normal isentropic flow exists behind the shock (if a second condensation is excluded),

$$\frac{\rho^{*'}}{\rho_0'} = \frac{\rho^*}{\rho_0} \left[\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \right]$$

$$\frac{p^{*'}}{p_0'} = \frac{p^*}{p_0} \left[\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \right]$$

therefore

$$\frac{\rho_0'}{\rho_0} = \frac{\rho^{*'}}{\rho^*} \neq 1 \quad (15a)$$

$$\frac{p_0'}{p_0} = \frac{p^{*'}}{p^*} \neq 1 \quad (15b)$$

The relations that have thus been found for $Q = Q_{\max}$, namely, $p_0' \neq p_0$, $\rho_0' \neq \rho_0$, and $F^{*'} > F^*$, will be proved for $Q \neq Q_{\max}$ also and at the same time will be more precisely expressed.

If p_0'/p_0 equals K and the calculations are started from the identity

$$K = \left(\frac{\frac{p_1}{p_0}}{\frac{p_2}{p_0'}} \right) \frac{p_2}{p_1}$$

by use of known gas-dynamic formulas and equation (12) the following expression is obtained:

$$K = \frac{p_o'}{p_o} = \frac{\left(1 - \frac{k-1}{k+1} x_1^2\right)^{\frac{1}{k-1}}}{\left(1 - \frac{k-1}{k+1} x_2^2\right)^{\frac{k}{k-1}}} \left(1 + x_1^2 - \frac{2k}{k+1} Q x_1 x_2\right) \quad (16)$$

In view of equation (7d), it is evident that the ratio p_o'/p_o is a function of Q and x_1 , that is, the ratio depends upon the quantity of heat q liberated by the condensation and upon the Mach number directly ahead of the shock. Thus the ratio can have the value of 1 only for certain special values of these variables. In order to obtain some basis for the determination of the possible range of values of K , the behavior of K with the variation of Q and x_1 will next be investigated.

With regard to Q (x_1 being held constant), it is sufficient to confine the investigation to the expression

$$\bar{K} = \left(1 - \frac{k-1}{k+1} x_2^2\right)^{-\frac{k}{k-1}} \left(1 + x_1^2 - \frac{2k}{k+1} Q x_1 x_2\right)$$

Then

$$\frac{\partial \bar{K}}{\partial Q} = \frac{2k}{k+1} \left(1 - \frac{k-1}{k+1} x_2^2\right)^{-\frac{2k-1}{k-1}} \left\{ \frac{\partial x_2}{\partial Q} \left[x_2 (1 + x_1^2) - x_1 Q (x_2^2 + 1) \right] - x_1 x_2 \left(1 - \frac{k-1}{k+1} x_2^2\right) \right\}$$

By application of equation (7b), the quantity in the brackets becomes 0 and therefore

$$\frac{\partial \bar{K}}{\partial Q} < 0$$

The ratio p_o'/p_o decreases (at a given value of w_1/a^*) with increasing Q . The maximum value of K is thus reached at the minimum value of

$Q = 1$. In this case, $a^* = a^*$ and $x_1 x_2 = 1$. Thus from equation (16) is obtained

$$K_{\max} = \left[x_1^{2k} \frac{1 - \frac{k-1}{k+1} x_1^2}{x_1^2 - \frac{k-1}{k+1}} \right]^{\frac{1}{k-1}} \quad (17a)$$

This formula is naturally the same as that applying to the ordinary normal compression shock for p_o'/p_o . The maximum value that K_{\max} can assume through the variation of x_1 is accordingly reached when $x_1 = 1$ with $K_{\max \max} = 1$. This may also be mathematically shown

$$\frac{dK_{\max}}{dx_1} = -\frac{2k}{k+1} K_{\max}^{2-k} \frac{x_1^{2k-1} (x_1^2 - 1)}{\left(x_1^2 - \frac{k-1}{k+1}\right)^2}$$

Because $1 \leq x_1 \leq \sqrt{\frac{k+1}{k-1}}$, the derivative is negative (except at the limits of the range of values of x_1 where it disappears). For $x_1 = 1$ there is thus a maximum value of K_{\max} (namely, $K_{\max} = 1$) and for $x_1 = \sqrt{\frac{k+1}{k-1}}$ a minimum value (namely, $K_{\max} = 0$), which at the same time is the smallest conceivable value for K (because p_o'/p_o cannot become negative). Hence

$$0 \leq \frac{p_o'}{p_o} \leq 1 \quad (18)$$

The ratio of stagnation pressure p_o'/p_o lies between 0 and 1 and assumes the second value only if the condensation shock is nonexistent. (Because in the derivation of this theorem the sign of

$\sqrt{(1 + x_1^2)^2 - 4Qx_1^2}$ plays no part, it is valid for both signs.) Accordingly it is permissible to describe K as a throttling factor as is customary in compression shock theory.

When $Q = Q_{\max}$, a minimum value of K is reached for the given value of x_1 .

$$K_{\min} = \frac{1 + x_1^2}{2} \left[\frac{k+1}{2} \left(1 - \frac{k-1}{k+1} x_1^2 \right) \right]^{\frac{1}{k-1}} \quad (17b)$$

This relation is the same as equation (15b). In this case, the velocity directly behind the shock w_2 is equal to the sonic velocity a^* . The inequality $1 \leq Q \leq \frac{1+x_1^2}{2x_1}$ corresponds to the inequality $K_{\max} \geq K \geq K_{\min}$.

Figure 4 shows the quantities K_{\max} and K_{\min} as functions of x_1 . When $x_1 = \sqrt{\frac{k+1}{k-1}}$ both quantities become equal to 0. However, in accordance with the remarks at the end of the last section, this result is only of theoretical interest.

Now, conversely, Q rather than x_1 will be considered as given and fixed. It is found that

$$\frac{\partial K}{\partial x_1} = \frac{2k}{k+1} \frac{\left(1 - \frac{k-1}{k+1} x_1^2\right)^{\frac{2-k}{k-1}}}{\left(1 - \frac{k-1}{k+1} x_2^2\right)^{\frac{2-k}{k-1}}} \left\{ \frac{\partial x_2}{\partial x_1} \left(1 - \frac{k-1}{k+1} x_1^2\right) [x_2 (1+x_1^2) - Q x_1 (1+x_2^2)] \right. \\ \left. + \left(1 - \frac{k-1}{k+1} x_2^2\right) (x_1^2 - 1) (Q x_2 - x_1) \right\}$$

By application of equation (7b), the quantity within the brackets becomes 0. Furthermore, because from equations (8) and (8a)

$$Q x_2 - x_1 = x_1 \left(\frac{w_2}{w_1} - 1 \right) < 0$$

for all values of $x_1 > 1$

$$\frac{\partial K}{\partial x_1} < 0$$

The ratio p_o'/p_o (at a given value of Q) decreases with increasing Mach number.

Thus according to this reasoning also, the throttling factor assumes the theoretical minimum value $K_{\min}^{(Q)} = 0$ corresponding to

$(x_1)_{\max} = \sqrt{\frac{k+1}{k-1}}$. At the smallest value of x_1 that is in agreement with the given value of Q , K becomes largest. According to equation (13)

$$\frac{1}{2} \left(\frac{1}{x_1} + x_1 \right) \geq Q$$

Because for $x_1 > 1$ the expression on the left side decreases with decreasing x_1 , the smallest possible value of Q corresponds to the smallest possible value of x_1 denoted by W . Accordingly, W is determined by the equation

$$\frac{1 + W^2}{2W} = Q$$

whereby

$$W = Q \pm \sqrt{Q^2 - 1}$$

If $\frac{1 + W^2}{2W}$ is substituted for Q , it becomes evident that (because $W > 1$) in order to create the identity

$$W = Q + \sqrt{Q^2 - 1}$$

if the root is understood to signify its absolute value. When this value is inserted in equation (16) (x_2 becoming equal to 1) the following is obtained:

$$K_{\max}(Q) = \frac{1 + (Q + \sqrt{Q^2 - 1})^2}{2} \left\{ \frac{k+1}{2} \left[1 - \frac{k-1}{k+1} (Q + \sqrt{Q^2 - 1})^2 \right] \right\}^{\frac{1}{k-1}} \quad (19)$$

The right side of equation (19) is formally the same as that of equation (17b) if x_1 is replaced by $W = Q + \sqrt{Q^2 - 1}$. The mathematical basis for this is that with a fixed value of x_1 the quantities Q_{\max} and x_1 are so related by the same equation as the quantities Q and $(x_1)_{\min} = W$ with a fixed value of Q , that in both cases the expression for K in equation (16) undergoes the same simplification. Directly behind the shock, sonic velocity exists in both cases; this velocity thus occurs in every case wherein either the maximum quantity

of heat is liberated at a given Mach number, or the quantity of heat being given, the condensation takes place at the lowest possible Mach number. Figure 5 shows $K_{\max}^{(Q)}$ as a function of Q .

Because it was assumed (see Introduction) that the specific gas constant R is not altered by the shock, an equation may be written

$$\frac{p_o}{\rho_o T_o} = \frac{p_o'}{\rho_o' T_o'}$$

From this equation and equation (2b) it follows that

$$\frac{\rho_o'}{\rho_o} = \frac{1}{Q^2} \frac{p_o'}{p_o} \leq 1 \quad (20)$$

because it is true that $Q \geq 1$ and $K \leq 1$. The ratio of the stagnation densities cannot be greater than 1.

Furthermore, because the loss of mass in the flow due to the precipitation of the condensed water is to be considered insignificant and disregarded, the following statement of continuity is valid:

$$F^* \rho^* a^* = F^* \rho^* a^*$$

By consideration of equations (15a) and (2a), the following equation is obtained:

$$F^{*'} = F^* \frac{\rho_o}{\rho_o'} \frac{1}{Q}$$

and because of equation (20)

$$F^{*'} = F^* Q \frac{p_o}{p_o'} \geq F^* \quad (21)$$

The second equality sign is valid only when $Q = 1$ and $p_o'/p_o = 1$, that is, when there is no shock at all. For the description of the flow behind the normal condensation shock, a larger minimum cross section must be specified.

Being a discontinuous, that is an irreversible, process, the condensation shock involves an increase in entropy. If the entropy per unit of mass of the flowing gas ahead of the shock is written as

$$s = c_p \ln T_0 - R \ln p_0$$

and that behind the shock is written as

$$s' = c_p \ln T_0' - R \ln p_0'$$

with the aid of equation (2b) the following equation is obtained:

$$\Delta s = s' - s = c_p \ln Q^2 - R \ln K \quad (22)$$

In accordance with equation (16) the following statement applies: The increase of entropy due to the normal condensation shock is a function of the Mach number at which the shock occurs and of the quantity of heat liberated. In the case of an ordinary normal compression shock, the first member on the right side of equation (22) is lacking; consequently, with a given throttling factor, the increase of entropy due to a compression shock is smaller than that due to a condensation shock.

FLOW PROPERTIES AT EXIT CROSS SECTION OF NOZZLE

The flow properties in the exit cross section F_E of a wind-tunnel nozzle are the same as those in the whole test section of the tunnel and are therefore of especial interest. All flow properties referring to F_E will be denoted by the subscript E; all flow properties that have undergone alteration due to a condensation shock will be denoted by a prime (except that those directly behind the shock will have the subscript 2 as previously). Behind the shock a normal isentropic flow exists to which the known gas-dynamic laws can be applied. These laws connect the conditions at any selected cross section F downstream of the shock with the conditions directly behind the shock; these conditions are therefore functions of $x_1 = w_1/a^*$ and Q. Hence F_E , x_1 , and Q are determinative of the conditions at the exit cross section F_E . If two of these quantities are held constant, the conditions at the exit cross section depend only upon the third quantity.

As previously shown, the quantity p_0/p_0' increases both with fixed Q and increasing x_1 (a transfer of the shock downstream) and with fixed x_1 (fixed shock site) and increasing Q. Therefore, if F_E and Q are held constant (or F_E and x_1), in accordance with equation (21), $F^{*'}$ and therefore $F^{*'} / F_E$ also increases with increasing x_1 (or with increasing Q). If, however, $F^{*'} / F_E$

increases, then in the region of supersonic flows (which is what is actually found behind the shock) p_E'/p_O' , ρ_E'/ρ_O , and T_E'/T_O' increase, while w_E'/a^* and the Mach number Ma_E' decrease.

With a fixed exit cross section F_E and a given value of F^* , the flow properties in the nozzle are determined.

If Q also is regarded as fixed in accordance with equations (2a) and (2b) neither a^* nor T_O' are altered. Thus w_E' decreases and T_E' increases with increasing x_1 . From the continuity rule, $\rho_E'w_E' = \text{constant}$, it follows furthermore that ρ_E' increases and from the equation of state $p_E'/\rho_E'T_E' = \text{constant}$ that p_E' also increases. Because Ma_1 increases with x_1 and q remains constant with Q (compare equation (3)), the conclusion is made that:

If in a nozzle the Mach number at which a certain specified quantity of heat of condensation is supplied to the flow is increased (that is, if the shock occurs further downstream), at the exit cross section the Mach number and the absolute value of the velocity are less but the pressure, density, and temperature are greater.

If in addition to F_E , the Mach number Ma_1 is considered (that is, x_1 also) as fixed, then according to equation (2b) T_O' increases with increasing Q and T_E' thus increases all the more. However, a^* also increases while w_E'/a^* decreases. Hence no general statement concerning w_E' is possible. This impossibility is also evident from another line of reasoning. Namely, if the exit cross section and the location of the shock coincide $w_E' = w_2$; according to equation (7d), w_2 (which equals $x_2 Q a^*$) is a decreasing function of Q . If on the contrary the exit cross section is assumed as infinitely large

$$w_E' = a^*Q \sqrt{\frac{k+1}{k-1}}$$

where w_E' is an increasing function of Q . In accordance with the continuity rule, the behavior of the density with increasing Q thus also depends upon the size of F_E , that is (at a given F^*), upon the Mach number for which the nozzle is constructed. In order to investigate the variation of p_E' as a function of Q , the calculations may again be started from the equation of state by writing

$$\rho_E' T_E' = \rho^* T_O' \frac{\rho_E'}{\rho^*} \frac{T_E'}{T_O'}$$

From $F_E \rho_E' w_E' = F^* \rho^* a^*$ is derived

$$\frac{\rho_E'}{\rho^*} = \frac{F^* a^*}{F_E w_E'}$$

and with $T_O' = T_O Q^2$ (compare equation (2b)) is obtained

$$\rho_E' T_E' = \rho^* T_O \frac{F^*}{F_E} Q^2 \frac{a^*}{w_E'} \frac{T_E'}{T_O'}$$

Because according to equation (2a), $Q a^* = a^*$, if T_E'/T_O' is also expressed in the known manner in terms of w_E'/a^*

$$\rho_E' T_E' = \rho^* T_O \frac{F^*}{F_E} Q \frac{\left[1 - \frac{k-1}{k+1} \left(\frac{w_E'}{a^*} \right)^2 \right]}{\frac{w_E'}{a^*}}$$

The equation of state thus assumes the form

$$p_E' = \text{constant} \frac{Q \left[1 - \frac{k-1}{k+1} \left(\frac{w_E'}{a^*} \right)^2 \right]}{\frac{w_E'}{a^*}}$$

Differentiation with respect to Q yields

$$\frac{\partial p_E'}{\partial Q} = \text{constant} \frac{\frac{w_E'}{a^*} \left[1 - \frac{k-1}{k+1} \left(\frac{w_E'}{a^*} \right)^2 \right] - Q \frac{\partial \left(\frac{w_E'}{a^*} \right)}{\partial Q} \left[1 + \frac{k-1}{k+1} \left(\frac{w_E'}{a^*} \right)^2 \right]}{\left(\frac{w_E'}{a^*} \right)^2}$$

Because w_E'/a^* is a decreasing function of Q , $\partial \left(\frac{w_E'}{a^*} \right) / \partial Q < 0$ and $\frac{\partial p_E'}{\partial Q} > 0$ where p_E' increases with Q .

The over-all result is as follows:

If in a nozzle, the site of the shock being fixed, the quantity of heat of condensation liberated increases, the Mach number at the exit cross section decreases while the pressure and the temperature

increase. The behavior of the density and of the absolute value of the velocity depends upon the Mach number for which the wind tunnel is built.

If a normal condensation shock occurs in a nozzle, the same statements apply. [NACA comment: It is believed that this sentence, completely redundant in the present form, was intended to read: "If an oblique condensation shock occurs...".] As a consequence of the shock, q increases from $q = 0$ to a value $q > 0$.

In the absence of a shock, a Mach number equal to the reduced Mach number that would exist at the exit cross section if a shock had occurred will occur at a point in the nozzle upstream of the exit cross section. The quotients p/p_0 , ρ/ρ_0 , T/T_0 , and w/a^* , which are clearly functions of the Mach number, must at that point have the same value as the corresponding exit cross section quotients p_E'/p_0' , ρ_E'/ρ_0' , T_E'/T_0' , and w_E'/a^* if a shock occurs. Hence it follows that

$$\frac{p_E'}{p} = \frac{p_0'}{p_0} < 1, \quad \frac{\rho_E'}{\rho} = \frac{\rho_0'}{\rho_0} < 1, \quad \frac{T_E'}{T} = \frac{T_0'}{T_0} = Q^2 > 1, \quad \text{and} \quad \frac{w_E'}{w} = \frac{a^*'}{a^*} = Q > 1$$

That is,

If in a nozzle, as a consequence of a normal condensation shock, the Mach number at the exit cross section is reduced, at the exit cross section the pressure and the density are lower and the temperature and the absolute velocity are higher than would correspond to the same Mach number in the absence of the shock (this Mach number would then occur at a cross section upstream of the exit cross section).

If the last two of the three quantities F_E , x_1 , and Q are held constant, that is, if the site of the shock and the quantity of heat of condensation liberated are regarded as fixed given values, then with increasing F_E pressure, density, and temperature naturally decrease, while the Mach number and the absolute velocity increase, for downstream from the shock a normal isentropic flow exists. Answering the question of how the quotients p_E'/p_E , ρ_E'/ρ_E , T_E'/T_E , Ma_E'/Ma_E , and w_E'/w_E behave as functions of F_E is not equally easy.

By consideration of the last of these quotients first, in accordance with equation (2a), the following equation is applicable:

$$\frac{w_E'}{w_E} = Q \frac{\frac{w_E'}{a^*'}}{\frac{w_E}{a^*}}$$

from which by formal development

$$\frac{\partial \left(\frac{w_E'}{w_E} \right)}{\partial F_E} = Q \frac{\frac{w_E}{a^*} \frac{\partial \left(\frac{w_E'}{a^{*'}} \right)}{\partial F_E} - \frac{w_E'}{a^{*'}} \frac{\partial \left(\frac{w_E}{a^*} \right)}{\partial F_E}}{\left(\frac{w_E}{a^*} \right)^2}$$

Introduction of the known general relation between F^*/F and w/a^* , a process of computation, which will here be omitted, leads to

$$\frac{d \left(\frac{w}{a^*} \right)}{dF} = \frac{1}{F} \frac{w}{a^*} \frac{1 - \frac{k-1}{k+1} \left(\frac{w}{a^*} \right)^2}{\left(\frac{w}{a^*} \right)^2 - 1}$$

The application of this relation yields

$$\frac{\partial \left(\frac{w_E'}{w_E} \right)}{\partial F_E} = \frac{1}{F_E} \frac{w_E'}{w_E} \left[\frac{1 - \frac{k-1}{k+1} \left(\frac{w_E'}{a^{*'}} \right)^2}{\left(\frac{w_E'}{a^{*'}} \right)^2 - 1} - \frac{1 - \frac{k-1}{k+1} \left(\frac{w_E}{a^*} \right)^2}{\left(\frac{w_E}{a^*} \right)^2 - 1} \right]$$

According to equation (21), $F^{*'}/F_E > F^*/F_E$ and therefore $w_E'/a^{*'} < w_E/a^*$ and the bracketed expression is thereby greater than 0. The quotient w_E'/w_E thus increases with F_E ; it assumes its least value if F_E coincides with the shock front cross section, and its greatest value if $F_E \rightarrow \infty$. In the first case, $\frac{w_E'}{w_E} = \frac{w_2}{w_1}$, and in the second case $\frac{w_E'}{a^{*'}} = \frac{w_E}{a^*} = \sqrt{\frac{k+1}{k-1}}$. By consideration of equation (8), the following equation is obtained:

$$Q \frac{x_2}{x_1} \leq \frac{w_E'}{w_E} \leq Q \quad (23)$$

The gas velocity at the exit cross section can increase up to Q times the original velocity.

The expression on the left side of equation (23) is equivalent to $w_2/w_1 < 1$ (compare equation (8a)); that on the right side is $Q > 1$. Equation (23) explains the fact already mentioned that,

assuming F^* to be given, upon the occurrence of a normal condensation shock the behavior of the velocity at the exit cross section depends upon the magnitude of this shock. The velocity decreases if the exit cross section decreases and increases if the exit cross section increases. This relation is evident. The shock effects so great an increase of pressure, density, and temperature that the velocity is first decreased. Because the flow contains a greater total amount of energy after the shock, with a sufficiently great expansion, that is, with a sufficiently marked decrease in the proportion of caloric energy, the velocity must finally become greater than it could have been without the shock.

In order to determine the exit cross section \tilde{F}_E at which the velocity would remain precisely the same, the calculation may be started from the expression

$$\frac{F^{*'}}{F^*} = \frac{\frac{F^{*'}}{F_E}}{\frac{F^*}{F_E}} = \frac{\frac{w_E'}{a^{*'}} \left[1 - \frac{k-1}{k+1} \left(\frac{w_E'}{a^{*'}} \right)^2 \right]^{\frac{1}{k-1}}}{\frac{w_E}{a^*} \left[1 - \frac{k-1}{k+1} \left(\frac{w_E}{a^*} \right)^2 \right]^{\frac{1}{k-1}}}$$

According to equation (21), $F^{*'}/F^* = Q/K$. By inserting the relation $w_E' = w_E = \tilde{w}_E$, the following equation is obtained:

$$\left(\frac{\tilde{w}_E}{a^*} \right)^2 = \frac{k+1}{k-1} \frac{Q^{2k} - Q^2 K^{k-1}}{Q^{2k} - K^{k-1}}$$

The quantity K is a function only of the fixed selected values x_1 and Q . Corresponding to \tilde{w}_E/a^* in accordance with the known relation a certain value of F^*/\tilde{F}_E exists from which \tilde{F}_E may be calculated.

Because $Q > 1$ and $K < 1$, $\left(\frac{\tilde{w}_E}{a^*} \right)^2 < \frac{k+1}{k-1}$, that is, \tilde{F}_E has a finite value.

From the continuity expression $\rho_E' w_E' = \rho_E w_E$ and equation (23) is obtained

$$\frac{1}{Q} \frac{x_1}{x_2} \geq \frac{\rho_E'}{\rho_E} \geq \frac{1}{Q} \quad (24)$$

In the cross section \tilde{F}_E , $\rho_E' = \rho_E$; upstream of that point,

$\rho_E' > \rho_E$, and downstream $\rho_E' < \rho_E$. In spite of the compressing effect of the condensation shock, the density at the exit cross section can be less than it would have been in the absence of the shock.

As a consequence of the energy equation

$$i_o' = i + q$$

the following equation may be written

$$\frac{w_E'^2}{2} + i_E' = \frac{w_E^2}{2} + i_E + q$$

where with $i = c_p T$ it follows that

$$\frac{w_E'^2}{2} \left(\frac{w_E'^2}{w_E^2} - 1 \right) = c_p T_E \left(1 - \frac{T_E'}{T_E} \right) + q$$

Because w_E'/w_E continuously increases with increasing F_E , T_E'/T_E must continuously decrease. The least value of this quotient is reached when $w_E'/w_E = Q$. For this case, from the preceding equation, if q is expressed in terms of Q in accordance with equation (3), the following equation is obtained:

$$\frac{w_E'^2}{2} (Q^2 - 1) + c_p T_E \left(\frac{T_E'}{T_E} - 1 \right) = i_o (Q^2 - 1)$$

Because

$$\frac{w_E'^2}{2} + c_p T_E = i_o$$

it follows from the previous equation that

$$\left(\frac{T_E'}{T_E} \right)_{\min} = Q^2$$

The maximum value for T_E'/T_E is T_2/T_1 (condensation shock occurring at the exit cross section). Therefore the following equation is obtained (compare equation (10)):

$$Q^2 \frac{1 - \frac{k-1}{k+1} x_2^2}{1 - \frac{k-1}{k+1} x_1^2} \geq \frac{T_E'}{T_E} \geq Q^2 \quad (25)$$

As may be seen, the statement is confirmed that, as the result of a normal condensation shock ($Q \neq 1$), the temperature at the exit cross section is raised.

For the ratio of the Mach numbers, the following relation is obtained:

$$\frac{Ma_E'}{Ma_E} = \frac{w_E'}{w_E} \frac{a_E}{a_E'} = \frac{w_E'}{w_E} \sqrt{\frac{T_E}{T_E'}}$$

Thus the ratio increases with F_E and as $F_E \rightarrow \infty$ the ratio approaches the value 1, in accordance with equations (23) and (25). The least value is that of Ma_2/Ma_1 . Accordingly, by use of equation (11) the following equation is obtained:

$$\frac{x_2}{x_1} \sqrt{\frac{1 - \frac{k-1}{k+1} x_1^2}{1 - \frac{k-1}{k+1} x_2^2}} \leq \frac{Ma_E'}{Ma_E} \leq 1 \quad (26)$$

This equation corroborates the fact that the normal condensation shock reduces the Mach number at the exit cross section.

The equation of state yields

$$\frac{p_E'}{p_E} = \frac{\rho_E'}{\rho_E} \frac{T_E'}{T_E}$$

The pressure ratio is, like ρ_E'/ρ_E and T_E'/T_E , a decreasing function of F_E . By use of equations (12), (24), and (25) the following equation is obtained:

$$\frac{1 + x_1^2 - \frac{2k}{k+1} Q x_1 x_2}{1 - \frac{k-1}{k+1} x_1^2} \geq \frac{p_E'}{p_E} \geq Q \quad (27)$$

It is evident that the normal condensation shock increases the pressure at the exit cross section.

In equations (23) to (27), x_2 may in all cases be clearly expressed in terms of x_1 and Q , in accordance with equation (7d).

The results previously derived are valid without change in respect to any nozzle cross section downstream of the shock F that may be chosen to substitute for F_E . Therefore the results may be summarized as follows: With increasing distance downstream of the site of a condensation shock, the ratios p'/p , ρ'/ρ , and T'/T decrease while Ma'/Ma and w'/w increase. The numerical values of the flow properties constituting the numerators and the values of those constituting the denominators vary in the same manner. Therefore it may be concluded that:

Downstream of a normal condensation shock, the pressure, the density, and the temperature decrease more sharply with increasing nozzle cross section and the Mach number and the velocity increase more sharply than in the absence of the shock.

Translation by Edward S. Shafer,
National Advisory Committee
for Aeronautics.

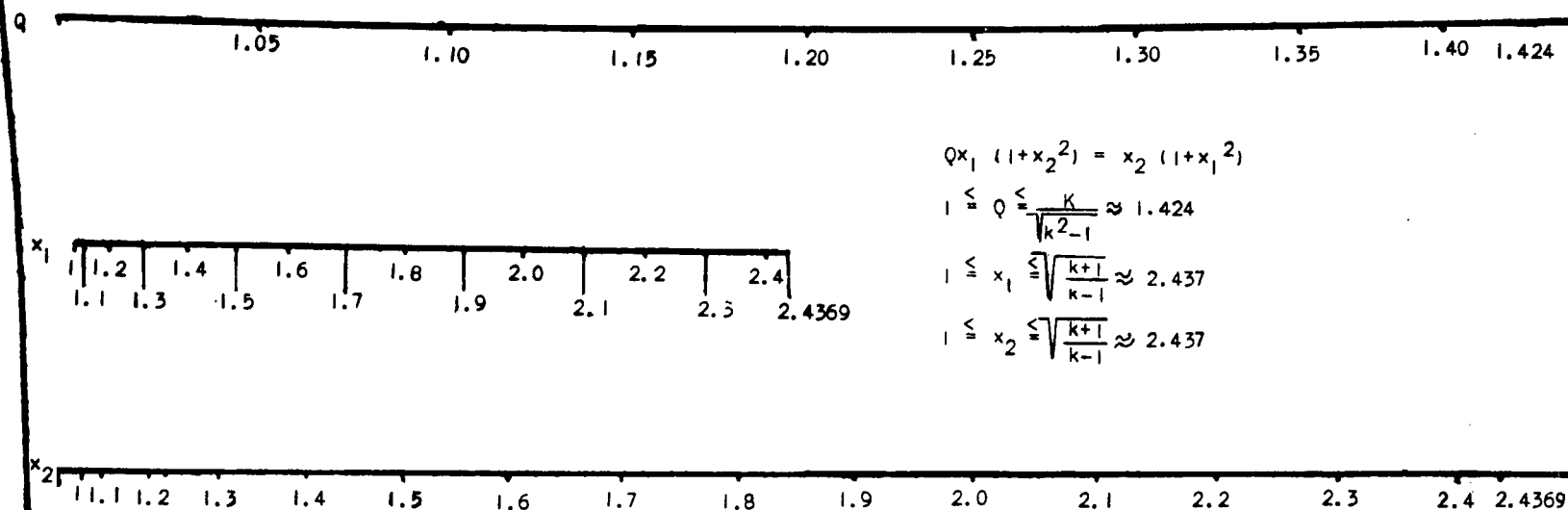


Figure 1.

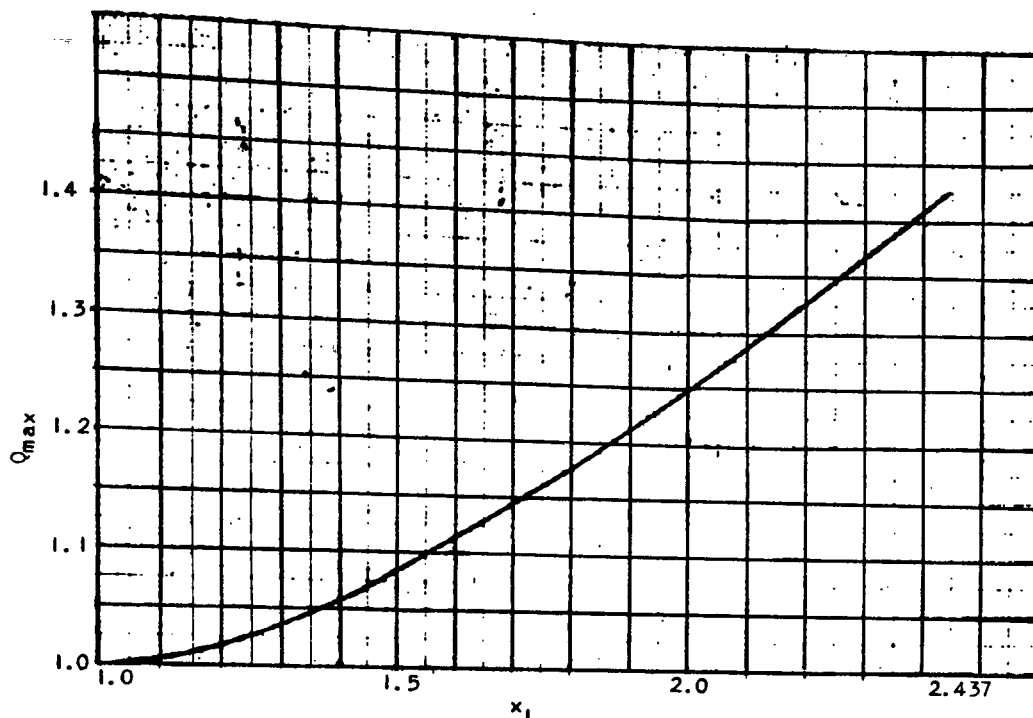


Figure 2.

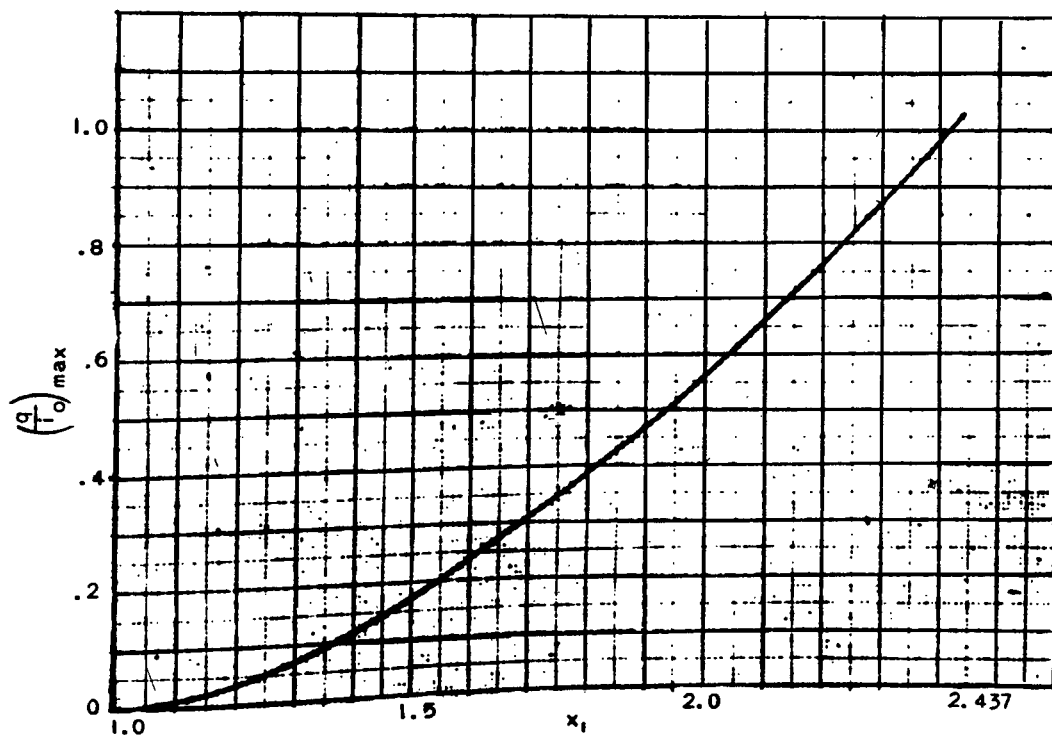


Figure 3.

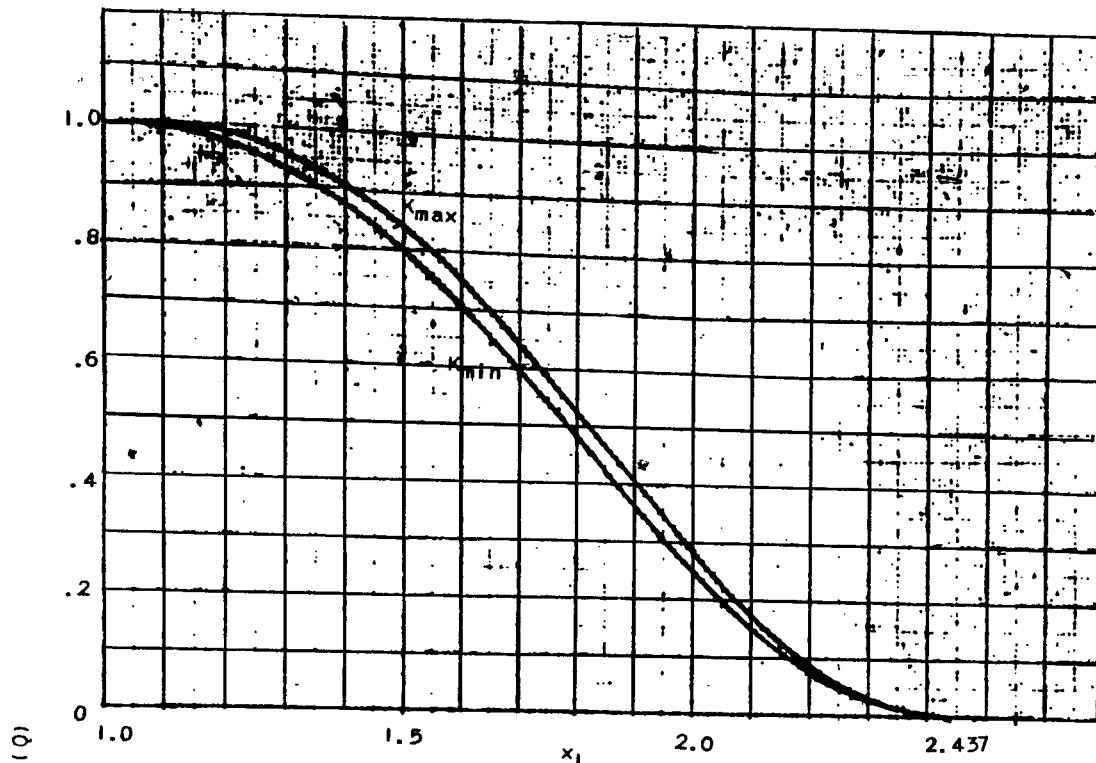


Figure 4.

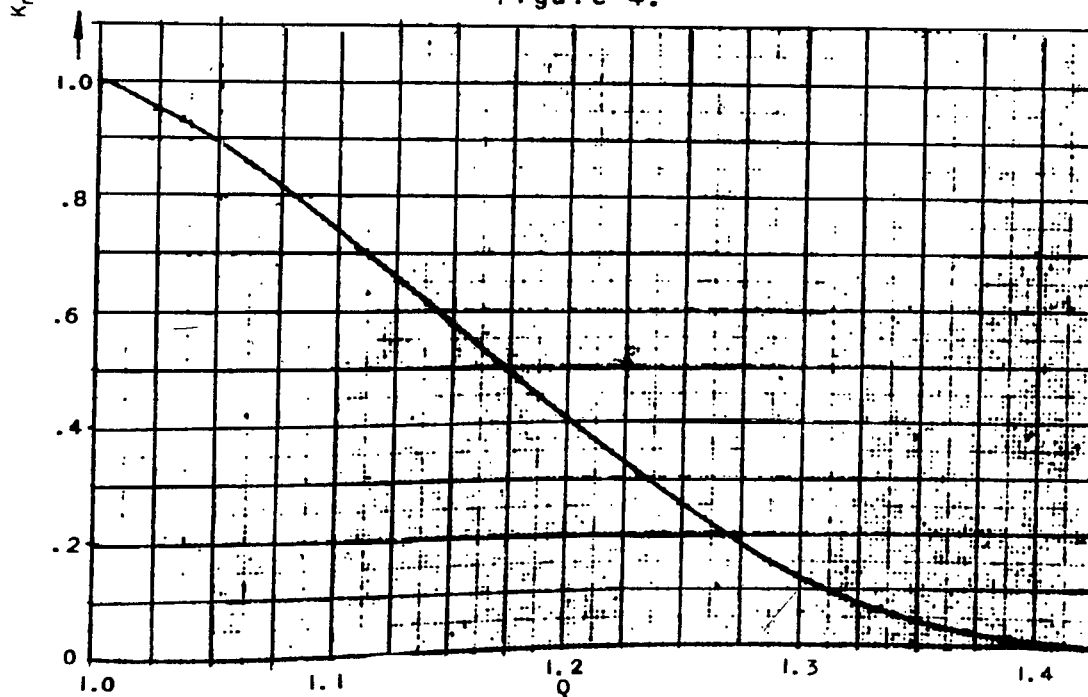


Figure 5.

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